13.4 Motion in Space: Velocity and Acceleration

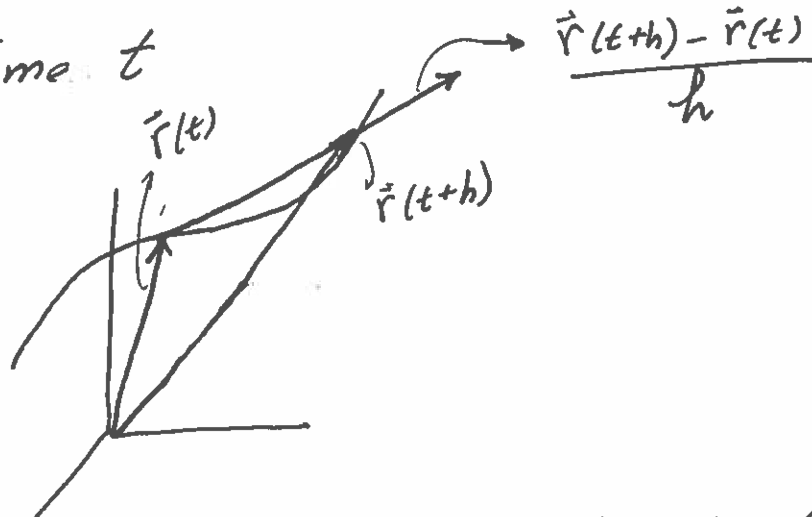
Consider a vector function

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle, \qquad t \in [a,b]$$

twice differentiable ($r'(t)$ and $r''(t)$ exist)

Assume that a particle in the space moves along the curve represented by $\vec{r}(t)$.

It means the particle is located at position $\vec{r}(t)$ at time t



<u>Distance travelled</u> by particle during the interval of length h:

$$|\vec{r}(t+h) - \vec{r}(t)|$$

Average speed during interval of time h:

$$\frac{|\vec{r}(t+h) - \vec{r}(t)|}{h}$$

Def 1.- The vector function

$$\frac{\vec{r}(t+h) - r(t)}{h}$$

is called "the average velocity vector over a time interval h."

Def 2.– $\vec{v}(t) = \vec{r}\,'(t) = \lim_{h \to 0} \dfrac{\vec{r}(t+h) - \vec{r}(t)}{h}$

is called "the velocity vector" at time t.

From previous def., we find that

velocity vector at t = tangent vector at t.

Def 3.– The magnitude of velocity vector

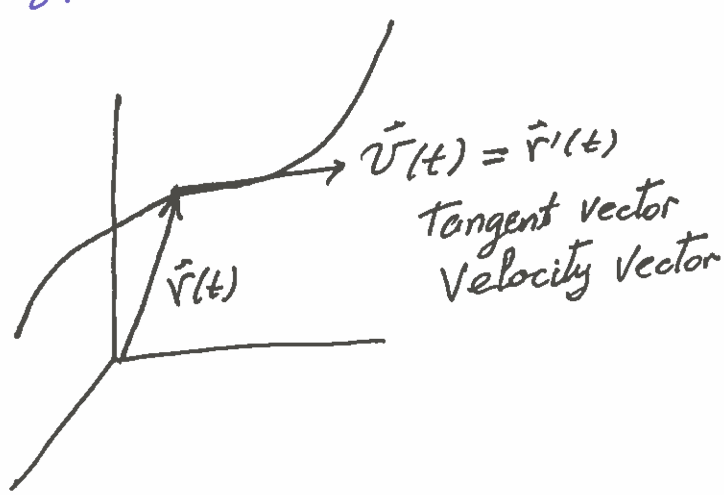
$|\vec{v}(t)| = |\vec{r}\,'(t)|$ is called the speed

of the particle at time t.

Def 4.– The derivative of the velocity vector

$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$ is called the acceleration

vector at t.



17) Find position $\hat{r}(t)$ of a particle which has

Acceleration: $\vec{a}(t) = 2t\,\hat{i} + \sin t\,\hat{j} + \cos 2t\,\hat{k}$

$\vec{v}(0) = \hat{i}$, $\hat{r}(0) = \hat{j}$
Initial Velocity, Initial position

$$\vec{v}'(t) = \vec{a}(t) \Rightarrow \vec{v}(t) = \int (2t\,\hat{i} + \sin t\,\hat{j} + \cos 2t\,\hat{k})\,dt =$$

$$= t^2\hat{i} - (\cos t)\,\hat{j} + \left(\frac{\sin 2t}{2}\right)\hat{k} + \vec{C}$$

Since $\hat{v}(0) = \hat{i}$

$$-\hat{j} + \hat{C} = \hat{i} \Rightarrow \hat{C} = \hat{i} + \hat{j} = \langle 1, 1, 0 \rangle$$

$$\Rightarrow \boxed{\vec{v}(t) = (t^2+1)\,\hat{i} - (\cos t - 1)\,\hat{j} + \frac{\sin 2t}{2}\hat{k}.}$$

$$\hat{r}(t) = \int \left((t^2+1)\,\hat{i} - (\cos t - 1)\,\hat{j} + \frac{\sin 2t}{2}\hat{k}\right)dt =$$

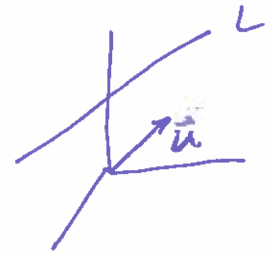
$$= \left(\frac{t^3}{3} + t\right)\hat{i} - (\sin t - t)\,\hat{j} - \frac{\cos 2t}{4}\hat{k} + \vec{d}$$

Since $\vec{r}(0) = \hat{j} \Rightarrow$

$$\hat{j} = -\frac{1}{4}\hat{k} + \vec{d} \Rightarrow \vec{d} = \hat{j} + \frac{1}{4}\hat{k} = \langle 0, 1, \tfrac{1}{4} \rangle$$

$$\Rightarrow \boxed{\vec{r}(t) = \left(\frac{t^3}{3} + t\right)\hat{i} - (\sin t - t - 1)\,\hat{j} - \frac{1}{4}(\cos 2t - 1)\,\hat{k}}$$

Motion along straight line



Equ. of a line in space

$$L : \vec{r}(t) = \vec{r}_0 + t\,\vec{u}$$

$\Rightarrow \quad \vec{r}\,'(t) = \vec{u}$. It means velocity vector is constant. $\vec{v}(t) = \vec{u}$

and $\vec{r}\,''(t) = \vec{0}$ No acceleration

This is called uniform motion along the line L.

If $\quad \vec{r}_2(t) = \vec{r}_0 + e^t\,\vec{u}$ what curve is this?

Same line!. what's the difference

$\vec{r}_2\,'(t) = e^t\,\vec{u} \neq$ Const. vector

$\vec{r}_2\,''(t) = e^t\,\vec{u} \Rightarrow$ acceleration $\neq \vec{0}$.

This is an accelerated motion along the same line L.

<u>Exercise</u>: For an object of mass m that moves in a circular path with constant angular speed ω has a position vector function:

$$\vec{r}(t) = a\cos\omega t\,\hat{i} + a\sin\omega t\,\hat{j}$$

Find the force acting on the object.

<u>Velocity</u>: $\vec{v}(t) = \vec{r}\,'(t) = -a\omega\sin\omega t\,\hat{i} + a\omega\cos\omega t\,\hat{j}$

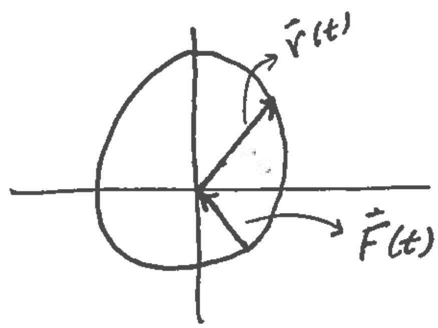
<u>acceleration</u>: $\vec{a}(t) = \vec{r}\,''(t) = -a\omega^2\cos\omega t\,\hat{i} - a\omega^2\sin\omega t\,\hat{j}$

Then - by Newton 2nd Law:

$$m\,\vec{a}(t) = \sum \text{forces}.$$

$$\vec{F}(t) = m\,\vec{a}(t) = -m\omega^2\left(a\cos\omega t\,\hat{i} + a\sin\omega t\,\hat{j}\right)$$

$$= -m\omega^2\,\vec{r}(t)$$



centripetal force.

Projectile thrown

#28) – Batter hits ball 3ft above ground toward the fence.

– The fence is 10 ft high and 400ft. from home plate.

– Ball leaves bat with speed 115 ft/s at an angle 50° above horizontal.

IS IT A HOME RUN ?

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answ: In General

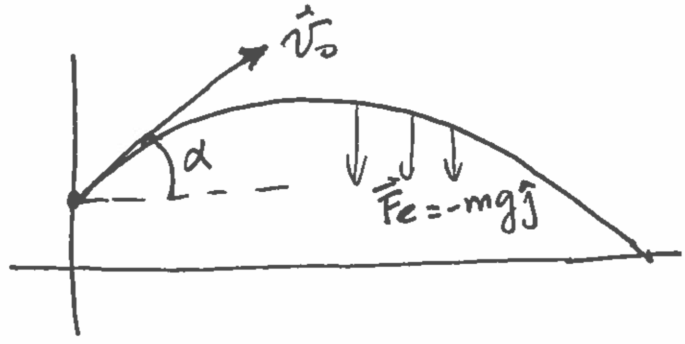
If angle of elevation is : α

Initial position : $\vec{r}(0) = \vec{r}_0$

Initial velocity : $\vec{V}(0) = \vec{V}_0$

Only external force is gravity : $F_e = -mg\,\hat{j}$

Find position vector at any time : $\underline{\vec{r}(t)}$



$$m\vec{a}(t) = \vec{F}_e = -mg\,\hat{j}$$

or $\boxed{m\,\vec{r}''(t) = -mg\,\hat{j}} \Rightarrow \vec{r}''(t) = -g\,\hat{j}$

$$\Rightarrow \quad \vec{v}(t) = \vec{r}'(t) = \;?$$

$$\int \vec{r}''(t)\,dt = \vec{r}'(t) + \vec{c} = -gt\,\hat{j} + \vec{c}$$

$$\Rightarrow \quad \vec{v}(t) = \vec{r}'(t) + \vec{c} = -gt\,\hat{j} + \vec{c}$$

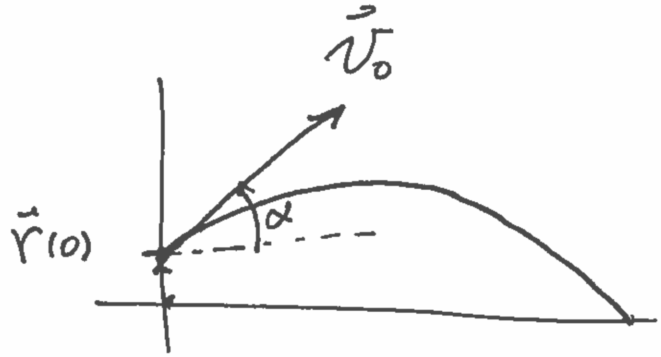
Since $\vec{v}(0) = \vec{v}_0 \Rightarrow \vec{c} = \vec{v}_0$

$$\Rightarrow \boxed{\vec{v}(t) = -gt\,\hat{j} + \vec{v}_0}$$

$$\Rightarrow \quad \vec{r}(t) = \int \vec{r}'(t)\,dt = \int \vec{v}(t)\,dt = -g\frac{t^2}{2}\,\hat{j} + \vec{v}_0 t + \vec{d}$$

Since $\vec{r}(0) = \vec{r}_0 \Rightarrow \vec{d} = \vec{r}_0$

$$\Rightarrow \boxed{\vec{r}(t) = -g\frac{t^2}{2}\,\hat{j} + \vec{v}_0 t + \vec{r}_0}$$



Then $\vec{v}_0 = |\vec{v}_0| \cos\alpha \, \hat{i} + |\vec{v}_0| \sin\alpha \, \hat{j}$

$$\Rightarrow \quad \vec{r}(t) = -g\frac{t^2}{2}\hat{j} + \left(|\vec{v}_0|\cos\alpha\right)t\,\hat{i} + \left(|\vec{v}_0|\sin\alpha\right)t\,\hat{j} + \vec{r}_0.$$

or

$$\boxed{\vec{r}(t) = \left(|\vec{v}_0|\cos\alpha\right)t\;\hat{i} + \left(\left(|\vec{v}_0 \sin\alpha\right)t - g\frac{t^2}{2}\right)\hat{j} + \vec{r}_0}$$

Tangential and Normal Components of Acceleration.

We want to express $\vec{a}(t)$ as

$$\boxed{\vec{a}(t) = \quad \vec{T}(t) + \quad \vec{N}(t)}$$

We know

$$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$$

Now, $\quad \vec{v}(t) = |\vec{v}(t)|\,\vec{T}(t)$

Since $\quad \vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \dfrac{\vec{v}(t)}{|\vec{v}(t)|}$

$$\Rightarrow \boxed{\vec{v}(t) = |\vec{v}(t)|\,\vec{T}(t)}.$$

$$\boxed{\vec{v}\,'(t) = |\vec{v}(t)|'\,\vec{T}(t) - |\vec{v}(t)|\,\vec{T}\,'(t)}$$

Now, $\quad \vec{T}\,'(t) = |T'(t)|\,\vec{N}(t)$

Since $\quad \vec{N}(t) = \dfrac{\vec{T}\,'(t)}{|T'(t)|} \Rightarrow \boxed{T'(t) = |\vec{T}\,'(t)|\,\vec{N}(t)}$

We can introduce the Curvature
dividing by $|\vec{r}'(t)|$ and multiplying by it.
$\quad\quad\quad\quad\quad\quad\quad\quad \| $
$\quad\quad\quad\quad\quad\quad\quad |\vec{v}(t)|$

It means

$$|\vec{v}(t)|\vec{T}'(t) = |\vec{v}(t)| \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} |\vec{r}'(t)| \vec{N}(t)$$

$$\overset{|\vec{v}(t)|=|\vec{r}'(t)|}{=\!=\!=} |\vec{v}(t)|^2 \kappa(t) \vec{N}(t)$$

Therefore,

$$\boxed{\vec{a}(t) = |\vec{v}(t)|\vec{T}(t) - \left(|\vec{v}(t)|^2 \kappa(t)\right)\vec{N}(t)}$$

or $\boxed{\vec{a}(t) = a_T(t)\,\vec{T}(t) + a_N(t)\,\vec{N}(t)}$

where $\boxed{a_T(t) = |\vec{v}(t)|'}$ and $\boxed{a_N(t) = |\vec{v}(t)|^2 \kappa(t)}$